MATH 245 F18, Exam 3 Solutions

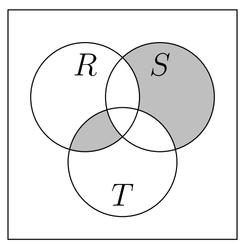
1. Carefully define the following terms: = (for sets), Associativity theorem (for sets), Distributivity theorem (for sets), De Morgan's Law (for sets).

   Two sets are equal if the have the exact same elements. Given sets R, S, T, the associativity theorem states that $(R \cup S) \cup T = R \cup (S \cup T)$, $(R \cap S) \cap T = R \cap (S \cap T)$, and $(R \Delta S) \Delta T) = R \Delta (S \Delta T)$. Given sets R, S, T, the distributivity theorem states that $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ and $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$. De Morgan's Law states, for sets R, S, U with $R \subseteq U$ and $S \subseteq U$, that $(R \cup S)^c = R^c \cap S^c$ and $(R \cap S)^c = R^c \cup S^c$.

2. Carefully define the following terms: power set, disjoint, equicardinal, relation.

   Given a set S, the power set of S is the set whose elements are all the subsets of S. Two sets are disjoint if their intersection is the empty set. Two sets are equicardinal if their elements can be paired off. Given sets S, T, a relation from S to T is a subset of $S \times T$.

3. Let R, S, T be sets. Draw a Venn diagram representing $(R \Delta S) \setminus (R \Delta T)$.



4. Let S, T be sets. Prove that $|S \times T| = |T \times S|$.

   Note: $|S \times T| = |S||T|$ is valid only for S, T finite.
   We pair the elements of $S \times T = \{(a, b) : a \in S, b \in T\}$ with the elements of $T \times S = \{(b, a) : b \in T, a \in S\}$ via $(a, b) \leftrightarrow (b, a)$, for all $a \in S$ and for all $b \in T$.

5. Let R, S, T be sets, with $S \subseteq T$. Prove that $R \cap S \subseteq R \cap T$.

   Let $x \in R \cap S$. Then $x \in R \wedge x \in S$, and by simplification twice we conclude both $x \in R$ and $x \in S$. Now, since $x \in S$ and $S \subseteq T$, we have $x \in T$. We apply conjunction to $x \in R$ and $x \in T$ to get $x \in R \wedge x \in T$. Lastly, this means that $x \in R \cap T$.

6. Let A, B be sets. Prove that $A \times (A \cap B) \subseteq (A \cup B) \times B$.

   Let $x \in A \times (A \cap B)$. Then, $x = (u, v)$ with $u \in A$ and $v \in A \cap B$. By addition, $u \in A \vee u \in B$, so $u \in A \cup B$. Now, $v \in A \wedge v \in B$, and by simplification $v \in B$. Hence $x = (u, v)$ with $u \in A \cup B$ and $v \in B$, so $x \in (A \cup B) \times B$.

7. Let S, T be sets with $T \subseteq S$. Let R be a transitive relation on S. Prove that $R|_T$ is transitive.

   Let $(a, b), (b, c) \in R|_T$. Then, $a, b, c \in T$ and also $(a, b), (b, c) \in R$. Since R is transitive, $(a, c) \in R$. Since $a, c \in T$, also $(a, c) \in R|_T$.

8. Let R, S, T, U be sets, with $R \subseteq U$ and $S \subseteq T \subseteq U$. Prove that $R \cup T^c \subseteq R \cup S^c$.

Let $x \in R \cup T^c$. Hence $x \in R \vee x \in T^c$. We now have two cases: $x \in R$ and $x \in T^c$.
Case $x \in R$: By addition, $x \in R \vee x \in S^c$. Hence, $x \in R \cup S^c$.
Case $x \in T^c$: Hence, $x \in U \setminus T$ and thus $x \in U \wedge x \notin T$. By simplification twice, we conclude both $x \in U$ and $x \notin T$. If $x \in S$, then (since $S \subseteq T$), $x \in T$, which is impossible. Thus $x \notin S$. We apply conjunction to $x \in U$ and $x \notin S$ to get $x \in U \wedge x \notin S$. Hence $x \in U \setminus S$, and thus $x \in S^c$. By addition, $x \in R \vee x \in S^c$ and hence $x \in R \cup S^c$.

9. Consider relation $S = \{(a, b) : a \leq b^2\}$ on $\mathbb{R}$. Prove or disprove that S is reflexive.

S is *not* reflexive. We need one explicit example, e.g. $0.5 \in \mathbb{R}$. Because $0.5 \not\leq 0.25 = (0.5)^2$, $(0.5, 0.5) \notin S$.

10. Consider relation $S = \{(a, b) : a \leq b^2\}$ on $\mathbb{R}$. Prove that $S^+ = R_{full}$.

In fact, we will prove $S \circ S = S^{(2)} = R_{full}$. Since $S^+ = S^{(1)} \cup S^{(2)} \cup$(other stuff), this will prove that $S^+ = R_{full}$. Let $a, b \in \mathbb{R}$ be arbitrary. Set $c = -\sqrt{|a|}$. Since $a \leq |a| = (-\sqrt{|a|})^2 = c^2$, we have $(a, c) \in S$. We also have $c = -\sqrt{|a|} \leq 0 \leq b^2$. Hence, $(c, b) \in S$. Combining $(a, c) \in S$ with $(c, b) \in S$, we conclude that $(a, b) \in S \circ S$.